

Polynomials Related to Harmonic Numbers and Evaluation of Harmonic Number Series I

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Abstract

In this paper we focus on two new families of polynomials which are connected with exponential polynomials $\phi_n(x)$ and geometric polynomials $F_n(x)$. We discuss their generalizations and show that these new families of polynomials and their generalizations are useful to obtain closed forms of some series related to harmonic numbers.

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1 Introduction

In this work we are interested in two new families of polynomials, namely *harmonic*–exponential polynomials and *harmonic*–geometric polynomials. We introduce these polynomials and discuss several interesting generalizations of them with the help of Theorem 1. Furthermore we list these polynomials and the numbers that we obtain from these polynomials.

Suppose we are given an entire function f and a function g , analytic in a region containing the annulus $K = \{z : r < |z| < R\}$ where $0 < r < R$. Hence these functions have following series expansions,

$$f(x) = \sum_{n=0}^{\infty} p_n x^n \text{ and } g(x) = \sum_{n=-\infty}^{\infty} q_n x^n.$$

The following theorem is related to the functions f and g .

Theorem 1 ([7]) *Let the functions f and g be described as above. If the series*

$$\sum_{n=-\infty}^{\infty} q_n f(n) x^n$$

converges absolutely on K , then

$$\sum_{n=-\infty}^{\infty} q_n f(n) x^n = \sum_{m=0}^{\infty} p_m \sum_{k=0}^m \left\{ \begin{matrix} m \\ k \end{matrix} \right\} x^k g^{(k)}(x) \quad (1)$$

holds for all $x \in K$.

Throughout this paper we consider the function g in Theorem 1 as a function which is analytic on the disk $K = \{z : |z| < R\}$. Hence the formula (1) turns out to be

$$\sum_{n=0}^{\infty} \frac{g^{(n)}(0)}{n!} f(n) x^n = \sum_{n=0}^{\infty} \frac{f^{(n)}(0)}{n!} \sum_{k=0}^n \left\{ \begin{matrix} n \\ k \end{matrix} \right\} x^k g^{(k)}(x). \quad (2)$$

We show that families of polynomials and their generalizations presented in this paper are considerably useful to obtain closed forms of some series related to harmonic numbers. For instance we obtain following closed forms:

$$\sum_{n=1}^{\infty} \left(\sum_{k=1}^n k H_k \right) x^n = \frac{x(1 - \ln(1-x))}{(1-x)^3}, \quad (3)$$

$$\sum_{n=1}^{\infty} \left(\sum_{k=1}^n k^2 H_k \right) x^n = \frac{x(1 + 2x - (1+x)\ln(1-x))}{(1-x)^4}. \quad (4)$$

Also for hyperharmonic series, one of our results is

$$\sum_{n=1}^{\infty} \left(\sum_{k=1}^n k H_k^{(\alpha)} \right) x^n = \frac{x(1 - \alpha \ln(1-x))}{(1-x)^{\alpha+2}} \quad (5)$$

where α is a nonnegative integer.

In the rest of this section we will introduce some important notions.

Stirling numbers of the first and second kind

Stirling numbers of the first kind $\left[\begin{matrix} n \\ k \end{matrix} \right]$ and Stirling numbers of the second kind $\left\{ \begin{matrix} n \\ k \end{matrix} \right\}$ are defined by means of ([1, 16])

$$(x)_n = x(x-1)\dots(x-n+1) = \sum_{k=0}^n \left[\begin{matrix} n \\ k \end{matrix} \right] x^k \quad (6)$$

and

$$x^n = \sum_{k=0}^n \left\{ \begin{matrix} n \\ k \end{matrix} \right\} (x)_k \quad (7)$$

respectively. These numbers are quite common in combinatorics ([4, 5, 12, 19]).

We note that for $n \geq k \geq 1$ the following identity holds for Stirling numbers of the second kind

$$\left\{ \begin{matrix} n \\ k \end{matrix} \right\} = \left\{ \begin{matrix} n-1 \\ k-1 \end{matrix} \right\} + k \left\{ \begin{matrix} n-1 \\ k \end{matrix} \right\}. \quad (8)$$

There is a certain generalization of these numbers namely r -Stirling numbers ([9]) which is similar to the weighted Stirling numbers ([10, 11]). Combinatorial meanings, recurrence relations, generating functions and several properties of these numbers are given in [9].

Exponential polynomials and numbers

Exponential polynomials (or single variable Bell polynomials) $\phi_n(x)$ are defined by ([2, 19])

$$\phi_n(x) := \sum_{k=0}^n \left\{ \begin{matrix} n \\ k \end{matrix} \right\} x^k. \quad (9)$$

Grunert stated these polynomials in terms of Stirling numbers of the second kind and obtained some fundamental formulas ([17]). Besides Grunert ([17]), mainly Ramanujan ([6]), Bell ([2]) and Touchard ([23]) are well-known studies on these polynomials. We refer [8] for comprehensive information on exponential polynomials.

The first few exponential polynomials are:

$\phi_0(x) = 1,$
$\phi_1(x) = x,$
$\phi_2(x) = x + x^2,$
$\phi_3(x) = x + 3x^2 + x^3,$
$\phi_4(x) = x + 7x^2 + 6x^3 + x^4.$

(10)

The well known exponential numbers (or Bell numbers) ([3, 12, 13]) are obtained by setting $x = 1$ in $\phi_n(x)$, i.e

$$\phi_n := \phi_n(1) = \sum_{k=0}^n \left\{ \begin{matrix} n \\ k \end{matrix} \right\}. \quad (11)$$

The first few exponential numbers are:

$$\phi_0 = 1, \phi_1 = 1, \phi_2 = 2, \phi_3 = 5, \phi_4 = 15. \quad (12)$$

Geometric polynomials and numbers

Geometric polynomials are defined in [21, 22] as follows:

$$F_n(x) := \sum_{k=0}^n \left\{ \begin{matrix} n \\ k \end{matrix} \right\} k! x^k. \quad (13)$$

We use F_n as one of the most common notations for these polynomials in the honor of Guido Fubini ([25]). These polynomials are also called as Fubini polynomials ([7]) or ordered Bell polynomials ([22]).

The first few geometric polynomials are:

$F_0(x) = 1,$
$F_1(x) = x,$
$F_2(x) = x + 2x^2,$
$F_3(x) = x + 6x^2 + 6x^3,$
$F_4(x) = x + 14x^2 + 36x^3 + 24x^4.$

(14)

Specializing $x = 1$ in (13) we get geometric numbers (or ordered Bell numbers) F_n as ([7, 22, 24]):

$$F_n := F_n(1) = \sum_{k=0}^n \left\{ \begin{matrix} n \\ k \end{matrix} \right\} k!. \quad (15)$$

According to ([25]), these numbers are called Fubini numbers by Comtet. The first few geometric numbers are:

$$F_0 = 1, F_1 = 1, F_2 = 3, F_3 = 13, F_4 = 75. \quad (16)$$

Boyadzhiev ([7]) introduced the "general geometric polynomials" as

$$F_{n,r}(x) = \frac{1}{\Gamma(r)} \sum_{k=0}^n \left\{ \begin{matrix} n \\ k \end{matrix} \right\} \Gamma(k+r) x^k, \quad (17)$$

where $\text{Re}(r) > 0$. In the third section we will deal with the general geometric polynomials.

Exponential and geometric polynomials are connected by the relation ([7])

$$F_n(z) = \int_0^\infty \phi_n(z\lambda) e^{-\lambda} d\lambda. \quad (18)$$

In [15] the authors obtained some fundamental properties of exponential and geometric polynomials and numbers using Euler- Seidel matrices method.

Harmonic and Hyperharmonic numbers

The n -th harmonic number is defined by the n -th partial sum of the harmonic series as

$$H_n := \sum_{k=1}^n \frac{1}{k}, \quad (19)$$

where $H_0 = 0$.

For an integer $\alpha > 1$, let

$$H_n^{(\alpha)} := \sum_{k=1}^n H_k^{(\alpha-1)} \quad (20)$$

with $H_n^{(1)} := H_n$, be the n -th hyperharmonic number of order α ([4, 13]).

These numbers can be expressed in terms of binomial coefficients and ordinary harmonic numbers as ([13, 18]):

$$H_n^{(\alpha)} = \binom{n+\alpha-1}{\alpha-1} (H_{n+\alpha-1} - H_{\alpha-1}). \quad (21)$$

Well-known generating functions of the harmonic and hyperharmonic numbers are given as

$$\sum_{n=1}^{\infty} H_n x^n = -\frac{\ln(1-x)}{1-x} \quad (22)$$

and

$$\sum_{n=1}^{\infty} H_n^{(\alpha)} x^n = -\frac{\ln(1-x)}{(1-x)^\alpha} \quad (23)$$

respectively ([14]).

The following relations connect harmonic and hyperharmonic numbers with the Stirling and r -Stirling numbers of the first kind as ([4]):

$$\begin{bmatrix} k+1 \\ 2 \end{bmatrix} = k! H_k, \quad (24)$$

and

$$k! H_k^{(r)} = \begin{bmatrix} n+r \\ r+1 \end{bmatrix}_r. \quad (25)$$

2 Transformation of harmonic numbers

In this section we study the series related to harmonic numbers using the transformation formula (2).

We set g in (2) as the generating function of harmonic numbers which is given by equation (22). After rearranging the k th derivative of the RHS of (22) we obtain the following nice looking result:

Proposition 2

$$\frac{d^k}{d^k x} \left\{ -\frac{\ln(1-x)}{1-x} \right\} = \frac{k! (H_k - \ln(1-x))}{(1-x)^{k+1}}. \quad (26)$$

Proof. Follows by induction on k . ■

From (26) we have

$$g^{(k)}(x) = \frac{k! (H_k - \ln(1-x))}{(1-x)^{k+1}} \quad (27)$$

and

$$g^{(k)}(0) = k! H_k. \quad (28)$$

Now we are ready to state a transformation formula for the series related to harmonic numbers.

Proposition 3 *For an entire function f the following transformation formula holds.*

$$\begin{aligned} \sum_{n=0}^{\infty} H_n f(n) x^n &= \frac{1}{1-x} \sum_{n=0}^{\infty} \frac{f^{(n)}(0)}{n!} \sum_{k=0}^n \begin{Bmatrix} n \\ k \end{Bmatrix} k! H_k \left(\frac{x}{1-x} \right)^k \\ &\quad - \frac{\ln(1-x)}{1-x} \sum_{n=0}^{\infty} \frac{f^{(n)}(0)}{n!} \sum_{k=0}^n \begin{Bmatrix} n \\ k \end{Bmatrix} k! \left(\frac{x}{1-x} \right)^k. \end{aligned} \quad (29)$$

Proof. Employing (27), (28) in (2) gives the statement. ■

Geometric polynomials $F_n(x)$ appear in the second part of the RHS of equation (29). The first part of the RHS contains a new family of polynomials. We will indicate them with $F_n^h(x)$ and call them as "harmonic-geometric polynomials" because of their factor H_k . Hence the harmonic-geometric polynomials are

$$F_n^h(x) := \sum_{k=0}^n \left\{ \begin{matrix} n \\ k \end{matrix} \right\} k! H_k x^k. \quad (30)$$

The first few harmonic-geometric polynomials are:

$F_0^h(x) = 0,$	(31)
$F_1^h(x) = x,$	
$F_2^h(x) = 3x^2 + x,$	
$F_3^h(x) = 11x^3 + 9x^2 + x,$	
$F_4^h(x) = 50x^4 + 66x^3 + 21x^2 + x,$	
$F_5^h(x) = 274x^5 + 500x^4 + 275x^3 + 45x^2 + x.$	

Using these notation we reformulate equation (29) as follows:

$$\sum_{n=0}^{\infty} H_n f(n) x^n = \frac{1}{1-x} \sum_{n=0}^{\infty} \frac{f^{(n)}(0)}{n!} \left\{ F_n^h\left(\frac{x}{1-x}\right) - F_n\left(\frac{x}{1-x}\right) \ln(1-x) \right\}. \quad (32)$$

Formula (32) enables us to calculate closed forms of some series related to harmonic numbers. Hence by means of (30) we give a corollary of Proposition 3.

Corollary 4 *For any nonnegative integer m the following equality holds.*

$$\sum_{n=1}^{\infty} n^m H_n x^n = \frac{1}{1-x} \left\{ F_m^h\left(\frac{x}{1-x}\right) - F_m\left(\frac{x}{1-x}\right) \ln(1-x) \right\}. \quad (33)$$

Proof. It follows directly by setting $f(x) = x^m$ in (32). ■

Remark 5 *Equation (33) is a generalization of the generating function of harmonic numbers, since the case $m = 0$ gives equation (22). Besides this ordinary case thanks to the formula (33) we obtain generating functions of several interesting series related to harmonic numbers. For instance the case $m = 1$ in (33) gives*

$$\sum_{n=1}^{\infty} n H_n x^n = \frac{x(1 - \ln(1-x))}{(1-x)^2}, \quad (34)$$

and the case $m = 2$ gives

$$\sum_{n=1}^{\infty} n^2 H_n x^n = \frac{x(1 + 2x - (1+x) \ln(1-x))}{(1-x)^3}, \quad (35)$$

and so on.

Now we extend our results to multiple sums.

Proposition 6 *We have*

$$\begin{aligned} \sum_{n=1}^{\infty} \left(\sum_{r=0}^n \binom{n+s-r}{s} r^m H_r \right) x^n &= \sum_{n=1}^{\infty} \left(\sum_{0 \leq i \leq i_1 \leq \dots \leq i_s \leq n} i^m H_i \right) x^n \\ &= \frac{1}{(1-x)^{s+2}} \left[F_m^h \left(\frac{x}{1-x} \right) - F_m \left(\frac{x}{1-x} \right) \ln(1-x) \right], \end{aligned} \quad (36)$$

where m and s are nonnegative integers.

Proof. By multiplying both sides of equation (33) with the Newton binomial series and considering

$$\sum_{r=0}^n \binom{n+s-r}{s} r^m H_r = \sum_{0 \leq i \leq i_1 \leq \dots \leq i_s \leq n} i^m H_i. \quad (37)$$

we obtain the statement. ■

Corollary 7

$$\sum_{n=0}^{\infty} H_n^{(s)} x^n = \frac{-\ln(1-x)}{(1-x)^s}.$$

Proof. Setting $m = 0$ and considering $F_0^h(x) = 0$ and $F_0(x) = 1$ gives the desired result. ■

As a one more corollary we have following graceful identity.

Corollary 8

$$\begin{aligned} \sum_{n=1}^{\infty} (1^m H_1 + 2^m H_2 + \dots + n^m H_n) x^n \\ = \frac{1}{(1-x)^2} \left[F_m^h \left(\frac{x}{1-x} \right) - F_m \left(\frac{x}{1-x} \right) \ln(1-x) \right]. \end{aligned} \quad (38)$$

Other values of m in (38) lead the following interesting sums:

$m = 1$ in (38) gives

$$\sum_{n=1}^{\infty} \left(\sum_{k=1}^n k H_k \right) x^n = \frac{x(1 - \ln(1-x))}{(1-x)^3}, \quad (39)$$

$m = 2$ gives

$$\sum_{n=1}^{\infty} \left(\sum_{k=1}^n k^2 H_k \right) x^n = \frac{x(1 + 2x - (1+x)\ln(1-x))}{(1-x)^4}, \quad (40)$$

and so on.

Remark 9 All these formulas and equations which we obtained until now, show that harmonic-geometric polynomials have strong relation with the series of harmonic numbers. We could state the generating functions of some series related to harmonic numbers in terms of harmonic-geometric polynomials, see for instance equations (33) and (36).

Remark 10 Most of the results in this section are obtained by setting $f(x) = x^m$ in the transformation formula (32). It is possible to obtain more general results by setting $f(x)$ in (32) as an arbitrary polynomial of order m as

$$f(x) = p_m x^m + p_{m-1} x^{m-1} + \cdots + p_1 x + p_0 \quad (41)$$

where $p_0, p_1, \dots, p_{m-1}, p_m$ are any complex numbers. Hence we get following equation which is more general than (33):

$$\begin{aligned} & \sum_{n=0}^{\infty} (p_m n^m + p_{m-1} n^{m-1} + \cdots + p_1 n + p_0) H_n x^n \\ &= \frac{1}{1-x} \sum_{k=0}^m p_k \left\{ F_k^h \left(\frac{x}{1-x} \right) - F_k \left(\frac{x}{1-x} \right) \ln(1-x) \right\}. \end{aligned} \quad (42)$$

Specializing coefficients of f gives more closed forms of harmonic number series. Each polynomial creates another sum. For instance by setting $p_k = 1$ for each $k = 0, 1, \dots, n$ in (41) we get

$$\begin{aligned} & \sum_{n=0}^{\infty} (n^m + n^{m-1} + \cdots + n + 1) H_n x^n \\ &= \frac{1}{1-x} \sum_{k=0}^m \left\{ F_k^h \left(\frac{x}{1-x} \right) - F_k \left(\frac{x}{1-x} \right) \ln(1-x) \right\}. \end{aligned} \quad (43)$$

This formula leads the following sums:

The case $m = 1$ in (43) gives

$$\sum_{n=1}^{\infty} (n+1) H_n x^n = \frac{x - \ln(1-x)}{(1-x)^2}. \quad (44)$$

The case $m = 2$ in (43) gives

$$\sum_{n=1}^{\infty} (n^2 + n + 1) H_n x^n = \frac{x^2 + 2x - (1+x^2) \ln(1-x)}{(1-x)^3}, \quad (45)$$

and so on.

By setting $p_k = k$ for each $k = 0, 1, \dots, n$ in (41) we get

$$\begin{aligned} & \sum_{n=1}^{\infty} (mn^m + (m-1)n^{m-1} + \cdots + n) H_n x^n \\ &= \frac{1}{1-x} \sum_{k=1}^m k \left\{ F_k^h \left(\frac{x}{1-x} \right) - F_k \left(\frac{x}{1-x} \right) \ln(1-x) \right\}. \end{aligned} \quad (46)$$

We can give some examples of special cases of (46) as well. For example the case $m = 1$ in (46) gives the sum (34). The case $m = 2$ in (46) gives

$$\sum_{n=1}^{\infty} n(2n+1) H_n x^n = \frac{3x(1+x) - x(x+3) \ln(1-x)}{(1-x)^3}, \quad (47)$$

and so on.

We obtain some of these results by using operator argument in the following subsection.

2.1 The operator (xD)

The operator (xD) operates a function $f(x)$ as

$$(xD) f(x) = x f'(x) \quad (48)$$

where f' is the first derivative of the function f .

For any m -times differentiable function f we have ([7]),

$$(xD)^m f(x) = \sum_{k=0}^m \left\{ \begin{matrix} m \\ k \end{matrix} \right\} x^k f^{(k)}(x). \quad (49)$$

This fact can be easily proven with induction on m using (8).

We consider the generating function of the harmonic numbers in the formula (49). With the help of Proposition 2 we have

$$\begin{aligned} (xD)^m \left(-\frac{\ln(1-x)}{1-x} \right) &= \sum_{k=0}^m \left\{ \begin{matrix} m \\ k \end{matrix} \right\} x^k \frac{k! (H_k - \ln(1-x))}{(1-x)^{k+1}} \\ &= \frac{1}{1-x} \left[F_m^h \left(\frac{x}{1-x} \right) - F_m \left(\frac{x}{1-x} \right) \ln(1-x) \right]. \end{aligned}$$

On the other hand by using (48) we have

$$(xD)^m \left(\sum_{n=1}^{\infty} H_n x^n \right) = \sum_{n=1}^{\infty} H_n n^m x^n.$$

Combining these two results we obtain the formula (33).

2.2 Harmonic-geometric numbers

Definition 11 *Harmonic-geometric numbers F_n^h are obtained by setting $x = 1$ in (30) as*

$$F_n^h := F_n^h(1) = \sum_{k=0}^n \left\{ \begin{matrix} n \\ k \end{matrix} \right\} k! H_k. \quad (50)$$

The first few *harmonic*-geometric numbers are

$$F_0^h = 0, F_1^h = 1, F_2^h = 4, F_3^h = 21, F_4^h = 138, F_5^h = 1095, \dots \quad (51)$$

Remark 12 By using (24) we can state *harmonic-geometric polynomials* and numbers just in terms of *Stirling numbers of the first and second kind* as follows

$$F_n^h(x) = \sum_{k=0}^n \left\{ \begin{matrix} n \\ k \end{matrix} \right\} \left[\begin{matrix} k+1 \\ 2 \end{matrix} \right] x^k, \quad (52)$$

$$F_n^h = \sum_{k=0}^n \left\{ \begin{matrix} n \\ k \end{matrix} \right\} \left[\begin{matrix} k+1 \\ 2 \end{matrix} \right]. \quad (53)$$

2.3 Harmonic-exponential polynomials and numbers

Geometric and exponential polynomials are connected to each other via equation (18). Now with this motivation we define *harmonic*-exponential polynomials and numbers.

Definition 13 *Harmonic-exponential polynomials and numbers are respectively given by the following equations*

$$\phi_n^h(x) := \sum_{k=0}^n \left\{ \begin{matrix} n \\ k \end{matrix} \right\} H_k x^k \quad (54)$$

and

$$\phi_n^h := \phi_n^h(1) = \sum_{k=0}^n \left\{ \begin{matrix} n \\ k \end{matrix} \right\} H_k. \quad (55)$$

The first few *harmonic*-exponential polynomials are,

$\phi_0^h(x) = 0,$	(56)
$\phi_1^h(x) = x,$	
$\phi_2^h(x) = x + \frac{3}{5}x^2,$	
$\phi_3^h(x) = x + \frac{9}{5}x^2 + \frac{11}{6}x^3,$	
$\phi_4^h(x) = x + \frac{21}{2}x^2 + 11x^3 + \frac{25}{12}x^4,$	
$\phi_5^h(x) = x + \frac{45}{2}x^2 + \frac{275}{6}x^3 + \frac{250}{12}x^4 + \frac{137}{60}x^5.$	

And *harmonic*-exponential numbers are,

$$\phi_0^h = 0, \phi_1^h = 1, \phi_2^h = \frac{5}{2}, \phi_3^h = \frac{22}{3}, \phi_4^h = \frac{295}{12}, \phi_5^h = \frac{1849}{20}, \dots \quad (57)$$

We can extend the relation (18) for harmonic types of these polynomials as

$$F_n^h(z) = \int_0^\infty \phi_n^h(z\lambda) e^{-\lambda} d\lambda. \quad (58)$$

3 Hyperharmonic-geometric and exponential polynomials

In this section we generalize almost all of our results which we obtained in the previous section.

Now instead of the generating function of harmonic numbers, we consider the generating function of the hyperharmonic numbers (23) in (2). Let us take g in (2) as

$$g(x) = \sum_{n=1}^{\infty} H_n^{(\alpha)} x^n = -\frac{\ln(1-x)}{(1-x)^\alpha}.$$

Next proposition gives a nice formula for k th derivatives of $g(x)$.

Proposition 14

$$\frac{d^k}{dx^k} \left\{ -\frac{\ln(1-x)}{(1-x)^\alpha} \right\} = \frac{\Gamma(k+\alpha)}{\Gamma(\alpha)} \frac{1}{(1-x)^{\alpha+k}} (H_{k+\alpha-1} - H_{\alpha-1} - \ln(1-x)). \quad (59)$$

Proof. Follows by induction on k . ■

Hence we have

$$g^{(k)}(x) = \frac{\Gamma(k+\alpha)}{\Gamma(\alpha)} \frac{1}{(1-x)^{\alpha+k}} (H_{k+\alpha-1} - H_{\alpha-1} - \ln(1-x)) \quad (60)$$

and

$$g^{(k)}(0) = \frac{\Gamma(k+\alpha)}{\Gamma(\alpha)} (H_{k+\alpha-1} - H_{\alpha-1}). \quad (61)$$

In the light of equation (21) we can state (61) simply as

$$g^{(k)}(0) = k! H_k^{(\alpha)}. \quad (62)$$

Now we are ready to prove the following proposition.

Proposition 15 *Let an entire function f be given. Then we have the following transformation formula:*

$$\begin{aligned} & \sum_{n=0}^{\infty} H_n^{(\alpha)} f(n) x^n \\ &= \frac{1}{(1-x)^\alpha} \sum_{n=0}^{\infty} \frac{f^{(n)}(0)}{n!} \sum_{k=0}^n \left\{ \begin{matrix} n \\ k \end{matrix} \right\} k! H_k^{(\alpha)} \left(\frac{x}{1-x} \right)^k \\ & \quad - \frac{\ln(1-x)}{(1-x)^\alpha} \sum_{n=0}^{\infty} \frac{f^{(n)}(0)}{n!} \frac{1}{\Gamma(\alpha)} \sum_{k=0}^n \left\{ \begin{matrix} n \\ k \end{matrix} \right\} \Gamma(k+\alpha) \left(\frac{x}{1-x} \right)^k \end{aligned} \quad (63)$$

Proof. Invoking (60) and (62) in (2) gives the statement. ■

The second part of the RHS of equation (63) contains the generalized geometric polynomials which are given by (17).

The first part of the RHS of equation (63) also contains a new family of polynomials which is a generalization of (30). We call this new family as "hyperharmonic-geometric polynomials". More clearly we give these polynomials as

$$F_{n,\alpha}^h(x) = \sum_{k=0}^n \left\{ \begin{matrix} n \\ k \end{matrix} \right\} k! H_k^{(\alpha)} x^k. \quad (64)$$

The first few hyperharmonic-geometric polynomials are:

$F_{n,\alpha}^h(x)$	$\alpha = 2$
$n = 0$	0
$n = 1$	x
$n = 2$	$x + 5x^2$
$n = 3$	$x + 15x^2 + 26x^3$
$n = 4$	$x + 35x^2 + 156x^3 + 154x^4$
$n = 5$	$x + 75x^2 + 650x^3 + 1540x^4 + 1044x^5$

(65)

and

$F_{n,\alpha}^h(x)$	$\alpha = 3$
$n = 0$	0
$n = 1$	x
$n = 2$	$x + 7x^2$
$n = 3$	$x + 21x^2 + 47x^3$
$n = 4$	$x + 49x^2 + 282x^3 + 342x^4$
$n = 5$	$x + 105x^2 + 1175x^3 + 3420x^4 + 2754x^5$

(66)

bear in mind that $F_{n,1}^h(x) = F_n^h(x)$ and we have a short list of these polynomials as (31).

In the case of $\alpha = 1$ equation (64) gives equation (30).

With the help of these notation we can write the transformation formula (63) simply as

$$\begin{aligned} & \sum_{n=0}^{\infty} H_n^{(\alpha)} f(n) x^n \\ &= \frac{1}{(1-x)^\alpha} \sum_{n=0}^{\infty} \frac{f^{(n)}(0)}{n!} \left[F_{n,\alpha}^h\left(\frac{x}{1-x}\right) - F_{n,\alpha}^h\left(\frac{x}{1-x}\right) \ln(1-x) \right] \end{aligned} \quad (67)$$

Now we give a plain formula as a corollary of Proposition 15.

Corollary 16

$$\sum_{n=1}^{\infty} n^m H_n^{(\alpha)} x^n = \frac{1}{(1-x)^\alpha} \left[F_{m,\alpha}^h\left(\frac{x}{1-x}\right) - F_{m,\alpha}^h\left(\frac{x}{1-x}\right) \ln(1-x) \right] \quad (68)$$

where m is a nonnegative integer.

Proof. Directly seen from the specializing $f(x) = x^m$ in (67). ■

Remark 17 Formula (68) is also a generalization of the generating function of hyperharmonic numbers since the case $m = 0$ gives (23).

Equation (68) also makes it possible to get closed forms of some series related to hyperharmonic numbers, for instance the case $m = 1$ in (68) gives

$$\sum_{n=1}^{\infty} n H_n^{(\alpha)} x^n = \frac{x(1 - \alpha \ln(1 - x))}{(1 - x)^{\alpha+1}} \quad (69)$$

where $F_{1,\alpha}(x) = \alpha x$, $F_{1,\alpha}^h(x) = x$.

Now we state a more general result which extends (68) to multiple sums.

Proposition 18

$$\begin{aligned} \sum_{n=1}^{\infty} \left(\sum_{r=0}^n \binom{n+s-r}{s} r^m H_r^{(\alpha)} \right) x^n &= \sum_{n=1}^{\infty} \left(\sum_{0 \leq i \leq i_1 \leq \dots \leq i_s \leq n} i^m H_i^{(\alpha)} \right) x^n \\ &= \frac{1}{(1-x)^{\alpha+s+1}} \left[F_{m,\alpha}^h \left(\frac{x}{1-x} \right) - F_{m,\alpha} \left(\frac{x}{1-x} \right) \ln(1-x) \right]. \end{aligned} \quad (70)$$

where m and s are nonnegative integers.

Proof. Multiplying both sides of equation (68) with the Newton binomial series and considering

$$\sum_{r=0}^n \binom{n+s-r}{s} r^m H_r^{(\alpha)} = \sum_{0 \leq i \leq i_1 \leq \dots \leq i_s \leq n} i^m H_i^{(\alpha)} \quad (71)$$

completes the proof. ■

Corollary 19

$$\sum_{n=0}^{\infty} H_n^{(s)} x^n = -\frac{\ln(1-x)}{(1-x)^s}$$

Proof. Specializing $m = 0$ and considering $F_{0,\alpha}^h(x) = 0$ and $F_{0,\alpha}(x) = 1$ gives statement. ■

For $m = 1$ we get the following corollary.

Corollary 20

$$\begin{aligned} \sum_{n=1}^{\infty} \left(\sum_{r=0}^n \binom{n+s-r}{s} r H_r^{(\alpha)} \right) x^n \\ = \sum_{n=1}^{\infty} \left(\sum_{0 \leq i \leq i_1 \leq \dots \leq i_s \leq n} i H_i^{(\alpha)} \right) x^n = \frac{x(1 - \alpha \ln(1-x))}{(1-x)^{\alpha+s+2}}. \end{aligned} \quad (72)$$

An example to the case $s = 0$ is

$$\sum_{n=1}^{\infty} \left(\sum_{k=1}^n k H_k^{(\alpha)} \right) x^n = \frac{x(1 - \alpha \ln(1 - x))}{(1 - x)^{\alpha+2}}. \quad (73)$$

The following gives a nice formula.

Corollary 21

$$\begin{aligned} & \sum_{n=1}^{\infty} \left(1^m H_1^{(\alpha)} + 2^m H_2^{(\alpha)} + \cdots + n^m H_n^{(\alpha)} \right) x^n \\ &= \frac{1}{(1 - x)^{\alpha+1}} \left[F_{m,\alpha}^h \left(\frac{x}{1 - x} \right) - F_{m,\alpha} \left(\frac{x}{1 - x} \right) \ln(1 - x) \right]. \end{aligned} \quad (74)$$

Remark 22 If we set $f(x)$ as an arbitrary polynomial of order m , such as

$$f(x) = p_m x^m + p_{m-1} x^{m-1} + \cdots + p_1 x + p_0 \quad (75)$$

where $p_0, p_1, \dots, p_{m-1}, p_m$ are any complex numbers, instead of $f(x) = x^m$ in (67) we obtain following general formula:

$$\begin{aligned} & \sum_{n=0}^{\infty} (p_m n^m + p_{m-1} n^{m-1} + \cdots + p_1 n + p_0) H_n^{(\alpha)} x^n \\ &= \frac{1}{(1 - x)^{\alpha}} \sum_{k=0}^m p_k \left\{ F_k^h \left(\frac{x}{1 - x} \right) - F_k \left(\frac{x}{1 - x} \right) \ln(1 - x) \right\}. \end{aligned} \quad (76)$$

Specializing (75) one can obtain several closed forms of hyperharmonic number series in a similar fashion to what we did after Remark 10.

3.1 Some results using the operator (xD)

For the completeness of this work we also consider these generalized arguments using operator (xD) .

If we set g as the generating function of hyperharmonic numbers (23) in (49), then we get

$$(xD)^m \left(-\frac{\ln(1 - x)}{(1 - x)^{\alpha}} \right) = \frac{1}{(1 - x)^{\alpha}} \left[F_{n,\alpha}^h \left(\frac{x}{1 - x} \right) - F_{n,\alpha} \left(\frac{x}{1 - x} \right) \ln(1 - x) \right].$$

On the other hand using (48) we have

$$(xD)^m \left(\sum_{n=1}^{\infty} H_n^{(\alpha)} x^n \right) = \sum_{n=1}^{\infty} H_n^{(\alpha)} n^m x^n.$$

Collecting these two results gives again equation (68)

$$\sum_{n=1}^{\infty} H_n^{(\alpha)} n^m x^n = \frac{1}{(1 - x)^{\alpha}} \left[F_{n,\alpha}^h \left(\frac{x}{1 - x} \right) - F_{n,\alpha} \left(\frac{x}{1 - x} \right) \ln(1 - x) \right].$$

3.2 Hyperharmonic-geometric numbers

Definition 23 *Hyperharmonic-geometric numbers $F_{n,\alpha}^h$ are obtained by setting $x = 1$ in (64) as*

$$F_{n,\alpha}^h := F_{n,\alpha}^h(1) = \sum_{k=0}^n \left\{ \begin{matrix} n \\ k \end{matrix} \right\} k! H_k^{(\alpha)}. \quad (77)$$

The first few *hyperharmonic-geometric* numbers are:

$F_{n,\alpha}^h$	$\alpha = 2$	$\alpha = 3$
$n = 0$	0	0
$n = 1$	1	1
$n = 2$	6	8
$n = 3$	42	69
$n = 4$	346	674
$n = 5$	3310	7455

(78)

bear in mind that $F_{n,1}^h = F_n^h$.

Remark 24 *Using (25) which is a relation between hyperharmonic numbers and r -Stirling numbers of the first kind, we can state hyperharmonic-geometric polynomials and numbers in terms of Stirling numbers as*

$$F_{n,r}^h(x) = \sum_{k=0}^n \left\{ \begin{matrix} n \\ k \end{matrix} \right\} \left[\begin{matrix} n+r \\ r+1 \end{matrix} \right]_r x^k, \quad (79)$$

and

$$F_{n,r}^h = \sum_{k=0}^n \left\{ \begin{matrix} n \\ k \end{matrix} \right\} \left[\begin{matrix} n+r \\ r+1 \end{matrix} \right]_r \quad (80)$$

respectively. One can easily see that the relations (79) and (80) are the generalizations of the relations (52) and (53).

Let us continue our work by defining a generalization of the exponential polynomials.

3.3 Hyperharmonic-exponential polynomials and numbers

Definition 25 *Hyperharmonic-exponential polynomials and numbers are defined respectively as*

$$\phi_{n,\alpha}^h(x) := \sum_{k=0}^n \left\{ \begin{matrix} n \\ k \end{matrix} \right\} H_k^{(\alpha)} x^k, \quad (81)$$

and

$$\phi_{n,\alpha}^h := \phi_{n,\alpha}^h(1) = \sum_{k=0}^n \left\{ \begin{matrix} n \\ k \end{matrix} \right\} H_k^{(\alpha)}. \quad (82)$$

The first few *hyperharmonic*–exponential polynomials are

$\phi_{n,\alpha}^h(x)$	$\alpha = 2$
$n = 0$	0
$n = 1$	x
$n = 2$	$x + \frac{5}{2}x^2$
$n = 3$	$x + \frac{15}{2}x^2 + \frac{13}{3}x^3$
$n = 4$	$x + \frac{35}{2}x^2 + 26x^3 + \frac{77}{12}x^4$
$n = 5$	$x + \frac{75}{2}x^2 + \frac{325}{3}x^3 + \frac{385}{6}x^4 + \frac{87}{10}x^5$

(83)

and

$\phi_{n,\alpha}^h(x)$	$\alpha = 3$
$n = 0$	0
$n = 1$	x
$n = 2$	$x + \frac{7}{2}x^2$
$n = 3$	$x + \frac{21}{2}x^2 + \frac{47}{6}x^3$
$n = 4$	$x + \frac{49}{2}x^2 + 47x^3 + \frac{57}{4}x^4$
$n = 5$	$x + \frac{105}{2}x^2 + \frac{1175}{6}x^3 + \frac{285}{2}x^4 + \frac{459}{20}x^5$

(84)

And also the first few *hyperharmonic*–exponential numbers are

$\phi_{n,\alpha}^h(x)$	$\alpha = 2$	$\alpha = 3$
$n = 0$	0	0
$n = 1$	1	1
$n = 2$	$\frac{7}{2}$	$\frac{9}{2}$
$n = 3$	$\frac{77}{6}$	$\frac{58}{3}$
$n = 4$	$\frac{611}{12}$	$\frac{347}{4}$
$n = 5$	$\frac{2197}{10}$	$\frac{24887}{60}$

(85)

Bearing in mind that, the case $\alpha = 1$ gives $\phi_{n,1}^h(x) = \phi_n^h(x)$ and $\phi_{n,1}^h = \phi_n^h$. The lists of $\phi_n^h(x)$ and ϕ_n^h in already given in (56) and (57).

Also for these new concepts we can generalize the relation (58) as

$$F_{n,\alpha}^h(z) = \int_0^\infty \phi_{n,\alpha}^h(z\lambda) e^{-\lambda} d\lambda. \quad (86)$$

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